Math 236: indirect proof

Please hand in your answers to these questions as part of the homework due 1/20

You are welcome to collaborate with your peers on these problems, or come ask me about them in office hours, or get help with them at the Skills Center. But please write your justifications up in your own words.

Methods of indirect proof:

I. *Proof by contrapositive*: to prove $P \Rightarrow Q$, prove the contrapositive $\sim Q \Rightarrow \sim P$. (This method takes advantage of the fact that the contrapositive is logically equivalent to the original implication, which we saw in Section 2.1.)

II. *Proof by contradiction*: to prove $P \Rightarrow Q$, assume that P is true and Q is false, and deduce a contradiction. This contradiction usually takes the form $R \wedge \sim R$ for some statement R.

Note that what is happening here is you're assuming that $\sim(P \Rightarrow Q)$ is true, and then deducing a contradiction, which shows that $\sim(P \Rightarrow Q)$ cannot be true, and hence $P \Rightarrow Q$ must be true.

If your contradiction is $Q \wedge \sim Q$, then most likely you've deduced Q from P, and you might be able to turn your proof into a shorter direct argument.

When to use indirect proof: Indirect proofs are particularly effective when the conclusion you're trying to prove has a negation that provides a natural starting point for an argument. For instance, the conclusion "a is irrational" has negation "a is rational". So to prove "a is irrational," you can start by assuming a is rational, i.e. $a = m/n$ for integers m and n with $n \neq 0$, and then proceed. See problem #1 for an example of this. There are several more examples in Section 2.2 of the text.

Exercises (there are 6 total, including some on the back):

1. Prove that if a real number x is irrational, then $\sqrt{x}$ is irrational.

[Hint: this statement has the form $P \Rightarrow Q$, where $\sim Q$ is "$\sqrt{x}$ is rational". Starting from there, you can either deduce $\sim P$ (i.e. "x is rational"), which will produce a proof by contrapositive, or you can add the assumption of P (i.e. "x is irrational") and deduce a contradiction, which will produce a proof by contradiction.]

2. In this exercise, we'll prove the eternally useful fact that for all sets A and B, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. Because this is an "if and only if" statement, we'll break it into two steps, proving each implication separately (recall that $P \Leftrightarrow Q$ is logically equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$).

Step 1: Let A and B be sets. Use a direct proof to prove that if $A = B$ then $A \subseteq B$ and $B \subseteq A$.

Step 2: Let A and B be sets. Use proof by contrapositive to prove that if $A \subseteq B$ and $B \subseteq A$ then $A = B$.

[Notes: proof by contrapositive is a nice choice in step 2 because it allows us to start with the rather clean hypothesis $A \neq B$, i.e. that A and B do not have exactly the same elements. This means that either some element of A is not in B, or vice versa.]

3. Prove that if n is an integer and 3 is a factor of n^2, then 3 is a factor of n.

[Notes: we say that an integer a is a factor of an integer b if there is an integer c with $ac = b$. You might try a proof by contrapositive on this problem. You may freely use a fact we will prove in Section 3.3: any integer can be written as one of $3k$, $3k+1$, or $3k+2$ for some $k \in \mathbb{Z}$.]

4. Prove that $\sqrt{3}$ is irrational.

[Notes: your proof should use the result of problem #3. You may use without proof that every natural number can be written as $3^r a$, where r is a natural number and a is not divisible by 3. If you get stuck, consult the proof in Example 4 on p. 62 for insipiration.]

5. Why does the proof in problem #4 break if $\sqrt{3}$ is replaced by $\sqrt{4}$?

6. Prove that $5\sqrt{3}$ is irrational [Hint: use the result of problem #4.]